German J Ophthalmol (1996) 5: 118–125

ORIGINAL ARTICLE

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A ray tracer for ophthalmological applications

Received: 25 October 1994
Accepted: 28 August 1995

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Abstract Ray tracing with a personal computer allows realistic simulation of optical properties of the human eye. Patterns of point sources are used as objects. The path of light rays is calculated between the point source and the retina for a Gullstrand eye model with improved parameters; the normal eye model has a resolution limit close to the natural resolution limit of the human eye. The image formed on the retina is projected back to a screen at the distance of the object so as to simulate image interpretation by the brain. Refractive errors are modeled by a change in eye parameters and corrected by eyeglasses or/and contact lenses or by an artificial intraocular lens. For optic correction the parameters of seeing aids can be fitted automatically by a least-squares routine. The effect of faulty eye correction on image quality is visualized by using a photograph of a realistic scene as an object.

Key words Ray tracing · Mathematical eye model · Tilted intraocular lenses · Shifted contact lenses · Optics

Zusammenfassung Die Berechnung der optischen Wege einzelner Lichtstrahlen gestattet eine realistische Simulation optischer Eigenschaften des menschlichen Auges. Punktmuster dienen als Objekt. Die Lichtstrahlen werden von der Lichtquelle bis zur Netzhaut für ein Gullstrand-Modellauge mit verbesserten Parametern berechnet; das Modellauge hat ein Auflösungsvermögen, welches dem natürlichen Auflösungsvermögen des menschlichen Auges nahekommt. Das auf der Netzhaut entstehende Bild wird auf einen Bildschirm in der Entfernung des Objekts zurückprojiziert, um die Bildinterpretation durch das menschliche Gehirn zu simulieren. Sehfehler werden durch Veränderung der Parameter des Modellauges modelliert und durch Brillen oder/und Kontaktlinsen oder implantierte Linsen korrigiert. Die Parameter von Sehhilfen werden nach der Methode der kleinsten Quadrate automatisch angepaßt. Der Effekt von fehlerhaften Sehhilfen auf die Bildqualität wird gezeigt durch Verwendung der Photographie einer realistischen Szene als Objekt.

Schlüsselwörter Strahldurchrechnung · Mathematisches Augenmodell · Verkippte Intraokularlinsen · Verschobene Kontaktlinsen · Optik

Introduction

The human eye is, among many other things, an optical instrument. Ophthalmologists should know the physics behind this instrument. However, what is, precisely, the physics behind the human eye? In first approximation the eye is a pinhole camera, in second approximation it is a camera with a thin lens and an aperture stop, in third approximation it has four cardinal points, in fourth approximation the photosensitive screen is nonplanar, and a few approximations higher one would have to solve second-order differential equations called eikonal equations. The latter task is impossible to achieve because the refracting field of a real eye is not given in terms of mathematical functions. In physics courses for medical students, teaching does not go beyond cardinal points, which means that it does not even go to a nonplanar screen. Clearly, there is an information gap!

Fortunately, there are computers, and modern personal computers really merit the term computers. What mathematics cannot achieve, namely, to illustrate physics, a computer can do by simulation. A student of physics or medicine has no problem imagining a single light ray: a straight line in a uniform medium, a change in direction at a refractive surface. Finding the precise formula for this change in direction was the great achievement of the Dutch mathematician Willebrord Snell van Rojen (better known as Snellius) 380 years ago. Since then, Snellius' law of refraction has become common sense. Every child who has pointed his flashlight at a swimming pool by night has developed a feeling for refraction. If we use a computer and a mathematical eye model we can calculate hundreds of light rays and show them on a monitor. We see bundles, we can imagine the (caustic) surface, and we see the waist of the bundles. We can check the influence of the aperture stop and the effect of focusing or defocusing on the diameter and position of the waist of the bundles. We get a vivid illustration of how the human eye forms an image and how refractive errors can impair the capacity of image formation.

The first purpose of the present paper is teaching the optics of the human eye by computer simulation. Our computer programs are deliberately kept simple. We use the eye model developed by Gullstrand [6]. We modify it a little but we stick to spherical surfaces. Very simple subroutines describe the straight-line sections of a light ray, find the intersection with an optical surface, and apply the law of Snellius. Then the calling program jumps back to calculate the next straight-line section, and the process is repeated until the light ray intersects the retina. All programs that contain physics are transparent to the reader. The graphics software, the menu, and the optimization routine are less transparent and may be considered black boxes, because one does not have to understand them to understand optical imaging. With nonspherical refractive surfaces, all programs would be black boxes.

The second goal of the present paper is to go beyond teaching and start with practical applications. We allow for deviations from the Gullstrand eye, e.g., myopia and hyperopia; allow for eyeglasses and/or contact lenses; and allow for artificial intraocular lenses. All lenses can be shifted away from the optical axes and/or tilted. At the present stage, however, and for the reason given above, we do not yet allow nonspherical surfaces.

We introduce another crude approximation. Whenever we deviate from a normal eye we have to ask what the patients will see. Nobody sees the image on their retina. They see an image at the position of the object. During baby- and childhood the brain develops an image interpretation "software" that correlates images on the retina with the outside world. Our present approximation is that this software has been developed with a normal Gullstrand eye and that it has never been modified.

Having mentioned two crude approximations, we must also mention two ingredients that are not trivial. First, we have modified Gullstrand's eye just as much as needed to make the ray-optical resolution limit at the fovea equal to the wave-optical resolution for photopic seeing. Second, we introduce a library routine for parameter fitting. Any one of the parameters that define the optical system, such as diopters of contact lenses or eyeglasses, can be marked as fitting parameters and any kind of image quality, such as good focusing at the fovea or good focusing at larger angles or a certain image size, can be defined as a fitting goal. The library routine will then vary the optical parameters until it finds a parameter set for which the sum of squared differences between the actual image and the goal becomes minimal.

Another purpose of the present paper is to help patients understand their problem. We want to show them what they will see after optical correction. This means that the computer takes a realistic scene, or a photograph, as an object and creates the images seen by an eye with various seeing aids.

Materials and methods

The eye model

In 1909 Gullstrand [6] introduced his famous eye model, which he called the *schematic eye*. It has six refractive surfaces: two for the cornea and four for a two-shell crystalline lens (see Fig. 1). All refractive surfaces as well as the retina have spherical shapes, the centers of the spheres being aligned along the optical axis. Gullstrand presented two sets of parameters: one set for an eye focused at the near point and one set for an eye focused at infinity. In Tables 1 and 2 we list Gullstrand's parameters. The spherical surfaces are defined by their radii together with the position of intersection with the optical axis; the front surface of the cornea defines a zero point, positions inside the eye are positive, and negative radii indicate concave surfaces. The notation is exemplified in Fig. 1.

Gullstrand's parameters are remarkably good, but not good enough for our purpose. To allow focusing and to improve resolu-

Table 1 The parameter set of the exact schematic eye presented by Gullstrand, focused at the near point

	Position [mm]	Radius [mm]	Refractive index
Cornea	0	7.7	1.376
	0.5	6.8	1.336
Eye lens	3.2	5.33	1.385
	3.8725	2.655	1.406
	6.5725	−2.655	1.385
	7.2	−5.33	1.336
Retina	24.0	−11.5	–

Table 2 The parameter set of the exact schematic eye presented by Gullstrand, focused at infinity

	Position [mm]	Radius [mm]	Refractive index
Cornea	0	7.7	1.376
	0.5	6.8	1.336
Eye lens	3.6	10.0	1.385
	4.146	7.911	1.406
	6.565	−5.76	1.385
	7.2	−6.0	1.336
Retina	24.0	−11.5	–

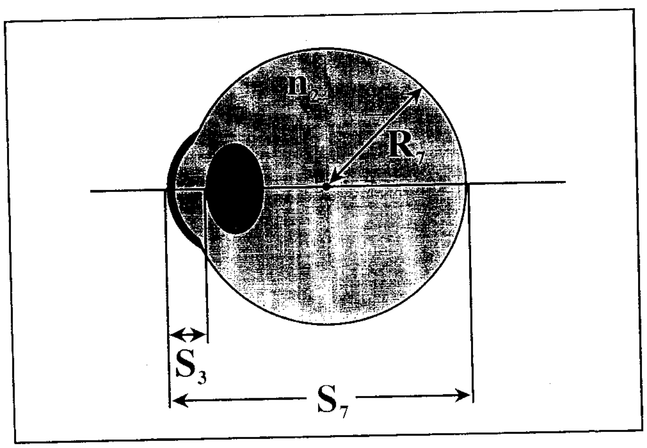

Fig. 1 The eye model presented by Gullstrand [6]; the parameters $S3$ and $S7$ for positions, $R7$ for radius, and n_2 and n_4 for indices of refraction exemplify the notation

tion we modify them. At any given distance of the object from the eye we take a superposition of the two parameter sets and let the computer do the focusing by optimizing a superposition parameter λ. Also, we introduce an iris with an adjustable aperture stop.

We get the following result. The ray-optical resolution limit of the modified Gullstrand eye for photopic seeing with an aperture stop of 2 mm comes out to be 1 min of angle at the fovea! This value approximates well the wave-optical resolution limit, which, in turn, is equal to the biological limit given by the packing density of cones at the fovea. For vision under angles of 5°, 10°, 20°, and 30° the ray-optical resolution limit of the modified Gullstrand eye becomes worse due to aberration effects in a rather realistic way.

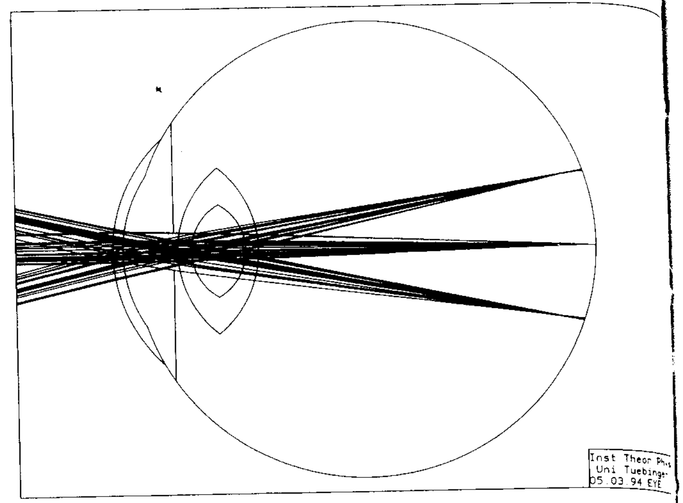

Fig. 2 Rays calculated by the ray tracer; only a small fraction of the calculated rays is shown

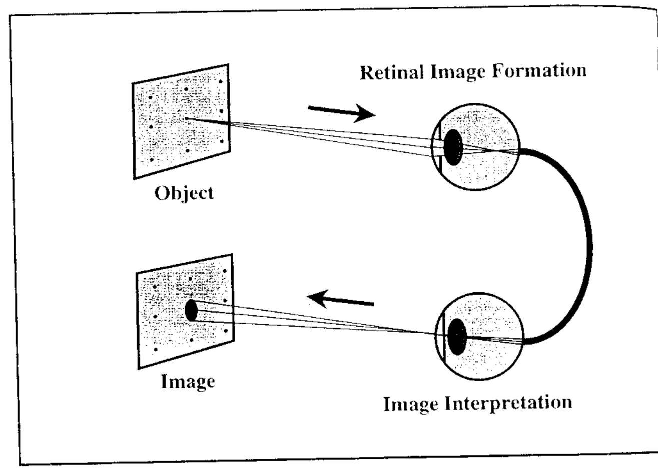

Fig. 3 Image formation and image interpretation; the image on the retina is only an intermediate piece of information

The present eye model allows simulation of normal vision in a realistic way and also allows simulation of ametropias. Myopia and hyperopia are created in the model by shifting of the position of the retina. We can modify the crystalline lens or replace it, for instance, by an implanted lens of given size and refractive power. We allow for eyeglasses of various shapes and power at various positions. We allow for contact lenses with or without a gap filled with tear film. We relax Gullstrand's condition of having all centers of spheres aligned along the optical axis. The seeing aids can be shifted away from the optical axis and can even be tilted. We do not allow, at this stage, nonspherical shapes. This also excludes cylindrical shapes.

Ray tracing

Ray tracing means following the path of a light ray. In our model the path of light is a straight line until it intersects a refractive surface. At the point of intersection, refraction takes place. The path chang-

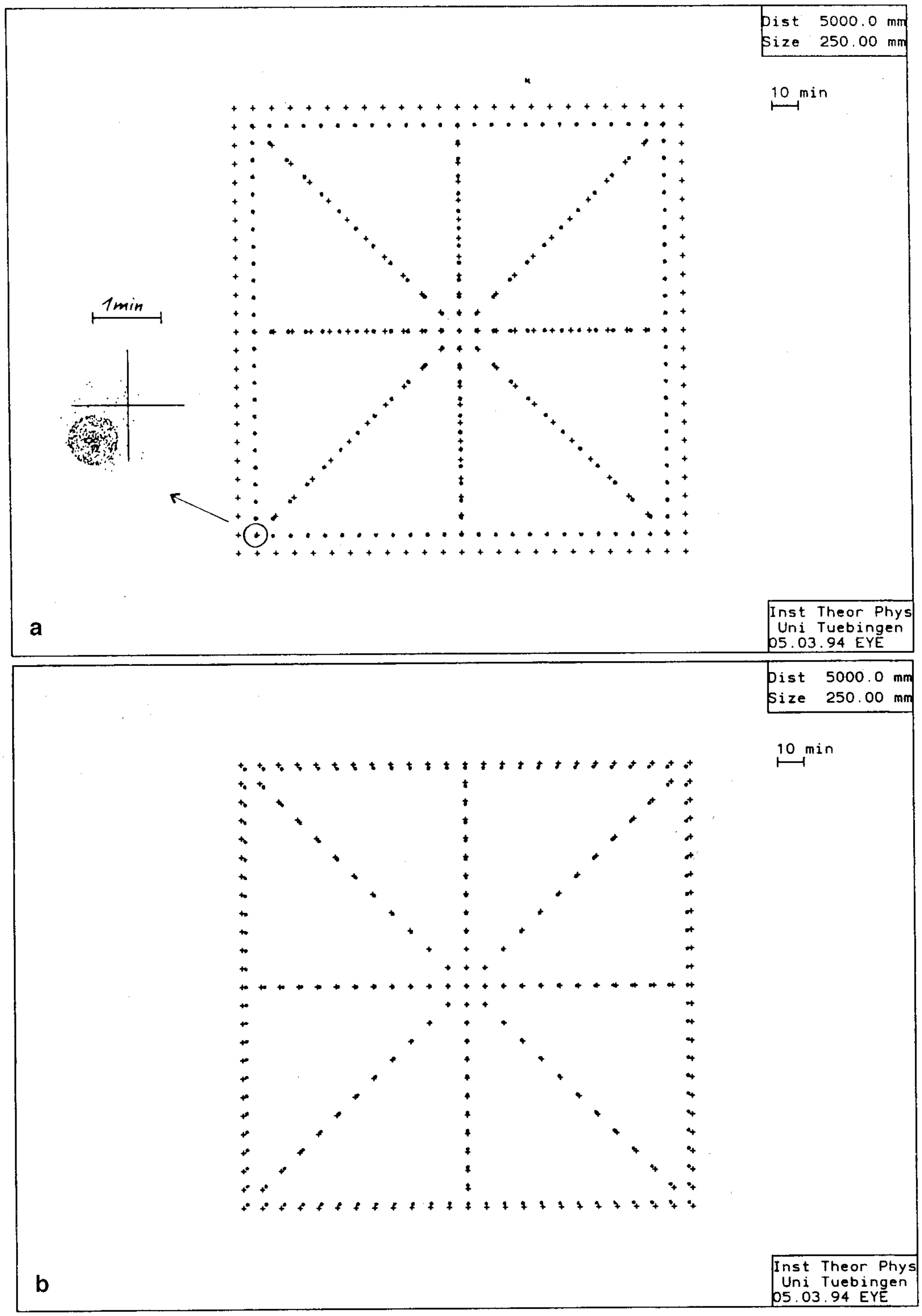

Fig. 4a, b Object and image calculated by the ray tracer. The object is a pattern of point sources (marked by *crosses*). The image consists of *spots* formed by rays intersecting the screen.
a Myopic eye corrected by eyeglass of −5.0 D.
b Same eye corrected by contact lens of −4.5 D

es direction and again becomes a straight line until it hits the next refractive surface. The change in direction is governed by the law of Snellius. The computer code starts with a straight line coming from some point of the object, calculates the position of the first intersection, applies Snellius' law, calculates the next point of intersection, and so on. The last point of intersection is the one at the retina. The procedure is repeated until a bundle of many light rays emerges from the given point of the object. The light rays of the bundle are chosen at random by a Monte Carlo routine. If a ray misses the aperture stop the computer discards it because it will not reach the retina. When the number of rays in a bundle reaches a given limit (usually several hundred) the computer will go to the next point of the object. For

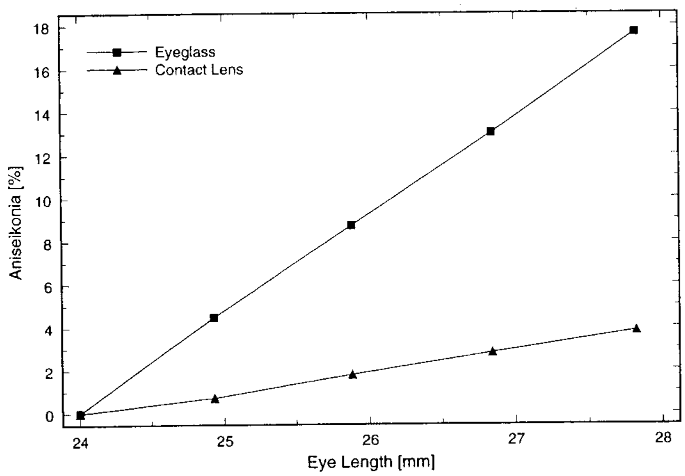

Fig. 5 Aniseikonia of myopic eyes corrected with eyeglasses or contact lenses; the aniseikonia is given as a percentage of the linear distortion relative to the normal eye

every point of the object we get a spot formed by the intersection of rays with the retina (see Fig. 2). All spots together define the image on the retina. A good description of ray tracing can be found in a textbook by Born et al. [2]. Applications in ophthalmology have been described elsewhere [1, 3–5, 8].

Image interpretation

At the retina, physics ends and image processing begins. We use a very simple model. We assume that a human being learns how to "see" at the time of baby- or childhood by associating images on the retina with objects that can be touched. We further assume that eyes are normal at that time, and we assume that the interpretation routine formed in the brain has never been modified.

From the assumptions we get the following model. From every intersection of an incoming ray with the retina we let an outgoing ray pass through the center of the iris of a normal eye until it reaches a screen at the position of the object. This outgoing ray marks a point on the screen. We get a number of points on the screen equal to the number of rays that have come from the object to the retina (see Fig. 3).

Note that for image interpretation we always use a normal eye and close the aperture stop down to a pinhole, even when the eye under investigation is not a normal eye. This means that the image on the screen will differ from the object. Even in the case in which the investigated eye is a normal Gullstrand eye, the image on the screen will show a spot of many points for every single point of the object.

Our model of image interpretation creates images at the position where we have the physiological impression of seeing an object. It thus simulates image processing of the human brain. The weak point of the model is the assumption that the interpretation routine in the brain has been formed with a normal eye and that this routine has never changed.

Objects

We use three types of objects, namely, a point pattern, a photograph, and a Landolt ring. The point pattern is depicted in Fig. 4. The crosses are not the object; rather, they only mark the position of infinitesimally small light sources. For every cross we get as an image a spot. For a normal Gullstrand eye with an aperture stop of 2 mm the diameter of the spot is approximately equal to the realistic resolution limit of the human eye. When the dot becomes larger the reason may be either a refractive error or a vision aid that is not ideal. The distance between spots and crosses can tell us something about distortion effects such as the prism effect of thick eyeglasses.

To get a vivid impression of what a patient with ametropia really sees, we use a realistic scene, or a photograph of a realistic scene, as an object. To determine the visual acuity we use Landolt rings of different sizes as objects.

Automatic focusing and optimization

We use a least-squares routine for focusing and for other kinds of optimization. We shall first consider focusing. An image spot, for instance a spot at the fovea, should become as small as possible. First, we calculate its center by the same formula that would be used to calculate its center of gravity. Next, we calculate the sum of squared distances between all the points that form the spot and the center of the spot. Then, we mark one or more parameters of the optical system such as, for instance, the power of an eyeglass or the parameter λ mentioned in the eye model. The marked parameter(s) will be varied by the least-squares routine until the sum of squared distances becomes minimal. The least-squares routine is a library routine [9, 10]. The routine implements a steepest-descent method with some special features, e.g., for saving computing time and avoiding traps [7, 11, 12]. For focusing we might want to make a compromise between the fovea, a 10° angle, and a 30° angle. In this case we add up the sum of squares of a spot at the fovea, a spot at 10°, and a spot at 30°, giving a higher weight to the spot at the center than to the peripheral ones.

Optimization by the least-squares routine is very flexible. We express the difference between an actual image and a desired image by a sum of squares, mark variation parameters, and call the least-squares routine. The routine will give us parameters that bring the image as close as possible to the desired one. An example is discussed below.

Results

The myopic eye

As a first application we investigate a myopic eye. We consider the case of axial myopia by enlarging the distance between the cornea and the retina in our eye model. The defect is corrected by eyeglasses or by contact lenses.

Figure 4 shows an example. In both cases the eye parameters are those of the normal eye except for the distance between the cornea and the retina, which has been enlarged from 24 to 25.9 mm. For correction with an eyeglass we need –5 D and for that with a contact lens, –4.5 D. These values have been determined automatically by the least-squares routine. The size of the spots shows the ray optical resolution. One of the dots is blown up 25 times to show how it is composed of 500 intersections of light rays with the retina. There is considerably less aniseikonia in the case of the contact lens. This feature is shown for various values of the eye length in Fig. 5.

The flexibility of the least-squares routine has been tested by offering an eyeglass together with a contact lens. The routine had to do the focusing *and* the compensati

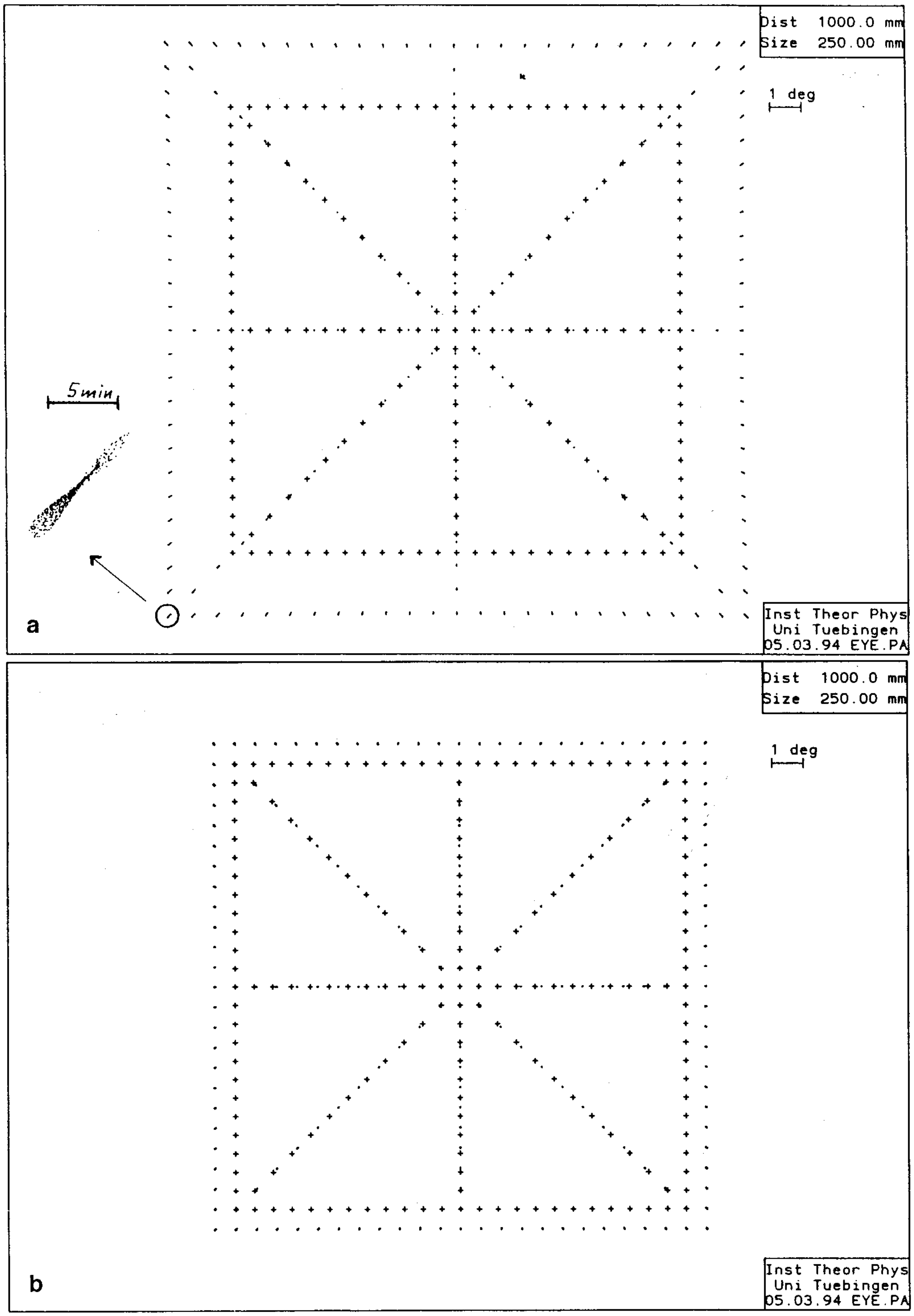

Fig. 6 Correction by eyeglass (**a**) and by contact lens (**b**) of a missing crystalline lens. The eyeglass has +11.75 D and the contact lens has +13.25 D. The combination of an eyeglass of −7.25 D with a contact lens of +19.75 D would completely eliminate the aniseikonia

of aniseikonia simultaneously by varying two parameters. It found +1.25 D for the eyeglass and −6.00 D for the contact lens. It should be noted that the routine found the principle of the Galilean telescope, or opera glass, without having any built-in knowledge of optics!

Correction of aphakia

Figure 6 shows the effect of eyeglass correction and contact-lens correction of aphakia. The eyeglass has +11.7 D and the contact lens has +13.25 D. A combination of an

Fig. 7 Image of a photograph seen and interpreted by the following eyes: **a** the normal Gullstrand eye, **b** a myopic eye with a –4.5-D contact lens displaced by 20°, **c** a myopic eye with a –5.0-D eyeglass tilted by 20°, and **d** an artificial intraocular lens tilted by 20°. The angle of 20° is unrealistic in all cases shown but clearly exhibits the distortion

eyeglass with –7.25 D and a contact lens with +19.75 D would eliminate the aniseikonia completely.

Decentered contact lenses, prism effect of eyeglasses, and dislocated and tilted artificial intraocular lenses

Figure 7 exhibits the effects of various eye modifications on images of a realistic scene. The object is a photograph. The image is composed of pixels like the ones on a color television screen. Details of Fig. 7 are given in the legend.

We should mention a technical detail. When the object is composed of point sources and the screen is continuous we proceed as described above in ray tracing. When the object is continuous and the screen has pixels, as in the present case, we let the computer calculate the path of light rays in the reverse direction. We start from a pixel, go to the retina (which is a continuous screen in our model), and proceed from there to the object. The computer then transfers the information bright/dark and color from the object to the pixel. Also, to economize on computer time we sacrifice some accuracy and take only around 40 rays/pixel instead of several hundred. Of course, the path of a light ray is the same when calculated in the reverse direction. Therefore, the imaging process may still be called ray tracing.

Discussion

By taking a linear superposition of two sets of parameters given by Gullstrand we obtained a model eye that has resolution limit comparable with that of the normal hum der photopic conditions. Ametropias (refractive errors) can be simulated by modifying of model parameters. Correcting eyeglasses and/or contact lenses may be added or an artificial lens can be inserted. All lenses can be shifted away from the optical axis and/or tilted.

A computer code that calculates images by ray tracing with many rays has been presented and several examples have been given for illustration. A simple model for image interpretation by the human brain has also been presented.

The main purpose of the present work is to aid teaching by computer simulation. Bundles of rays exhibit the process of retinal image formation in an illustrative way.

The present computer code may also be considered a first step toward application. Image qualities can be defined as a sum of squared deviations between the actual image and a desired ideal image. A least-squares fit of optical parameters will then be performed automatically.

The code consists of well-documented FORTRAN subroutines called by a TURBO PASCAL main program. A menu allows easy access. The least-squares fitting routine is taken from Moré et al. [12]. The code is available free of charge for educational, private, and research purposes only. (Interested readers should send a short message to *wolfgang.fink@uni-tuebingen.de* to get the address of the ftp-server, which provides the code.)

References

1. Bennet AG (1986) Two simple calculating schemes for use in ophthalmic optics. Ophthalmic Physiol Opt 6:419–429
2. Born M, Wolf E (1975) Principles of optics, 5th edn. Pergamon, Oxford
3. Fechner PU (1984) Intraokularlinsen: Grundlagen und Operationslehre, 2nd edn. Enke, Stuttgart, pp 47–71
4. Gernet H, Osthold H (1973) Augenseitige Optik: Ein neues Gebiet der Okulometrie. Ophthalmologica 166:120–143
5. Gernet H, Zörkendörfer S (1982) Zur Brechkraft von menschlichen Augenlinsen und von Kunstlinsen. Fortschr Ophthalmol 79:184–186
6. Gullstrand A (1909) Handbuch der physiologischen Optik, 3rd edn, vol 1. L. Voss, Hamburg Leipzig, p 226
7. Hebden MD (1973) An algorithm for minimization using exact second derivatives. Atomic Research Establishment report TP 515. Harwell, England
8. Kashiwagi T, Khu PM (1991) Ray tracing error correction in ophthalmic optics. J Cataract Refract Surg 17:194–198
9. Levenberg K (1944) A method for the solution of certain nonlinear problems in least squares. Quart Appl Math 2:164–168
10. Marquardt DW (1963) An algorithm for least-squares estimation of nonlinear parameters. SIAM J Appl Math 11:431–441
11. Moré JJ (1977) The Levenberg-Marquardt algorithm: implementation and theory, numerical analysis. In: Watson GA (ed) Lecture notes in mathematics, vol 630. Springer, Berlin Heidelberg New York, pp 105–116
12. Moré JJ, Garbow SB, Hillstrom KE (1980) User guide for MINPACK-1. Argonne National Laboratory, Applied Mathematics Division, ANL-80-74, Argonne, Illinois